Estimating the Implied Risk Neutral Density for the U.S. Market Portfolio

Extracting the Risk Neutral Density from Options Prices, in Theory

f(x) = risk neutral probability density function (RND)

F(x) = $\int_{-\infty}^{x} f(z)\,dz$ = risk neutral distribution function.

$$\frac{\partial C}{\partial X} = -e^{-rT}\int_X^{\infty} f(S_T)\,dS_T = -e^{-rT}\left[1 - F(X)\right]$$

$$F(X) = e^{rT}\frac{\partial C}{\partial X} + 1$$

$$f(X_n) \approx e^{rT}\frac{C_{n+1} - 2C_n + C_{n-1}}{(\Delta X)^2}.$$

Table 1
S&P 500 Index Options Prices, Jan. 5, 2005

S&P 500 Index closing level, = 1183.74 Interest rate = 2.69
Option expiration: 3/18/2005 (72 days) Dividend yield = 1.70

	Calls				Puts			
Strike price	Best bid	Best offer	Average price	Implied volatility	Best bid	Best offer	Average price	Implied volatility
500	-	-	-	-	0.00	0.05	0.025	0.593
550	-	-	-	-	0.00	0.05	0.025	0.530
600	-	-	-	-	0.00	0.05	0.025	0.473
700	-	-	-	-	0.00	0.10	0.050	0.392
750	-	-	-	-	0.00	0.15	0.075	0.356
800	-	-	-	-	0.10	0.20	0.150	0.331
825	-	-	-	-	0.00	0.25	0.125	0.301
850	-	-	-	-	0.00	0.50	0.250	0.300
900	-	-	-	-	0.00	0.50	0.250	0.253
925	-	-	-	-	0.20	0.70	0.450	0.248
950	-	-	-	-	0.50	1.00	0.750	0.241
975	-	-	-	-	0.85	1.35	1.100	0.230
995	-	-	-	-	1.30	1.80	1.550	0.222
1005	-	-	-	-	1.50	2.00	1.750	0.217
1025	-	-	-	-	2.05	2.75	2.400	0.208
1050	134.50	136.50	135.500	0.118	3.00	3.50	3.250	0.193
1075	111.10	113.10	112.100	0.140	4.50	5.30	4.900	0.183
1100	88.60	90.60	89.600	0.143	6.80	7.80	7.300	0.172
1125	67.50	69.50	68.500	0.141	10.10	11.50	10.800	0.161
1150	48.20	50.20	49.200	0.135	15.60	17.20	16.400	0.152
1170	34.80	36.80	35.800	0.131	21.70	23.70	22.700	0.146
1175	31.50	33.50	32.500	0.129	23.50	25.50	24.500	0.144
1180	28.70	30.70	29.700	0.128	25.60	27.60	26.600	0.142
1190	23.30	25.30	24.300	0.126	30.30	32.30	31.300	0.141
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1215	12.90	14.50	13.700	0.122	44.60	46.60	45.600	0.138
1220	11.10	12.70	11.900	0.120	47.70	49.70	48.700	0.136
1225	9.90	10.90	10.400	0.119	51.40	53.40	52.400	0.137
1250	4.80	5.30	5.050	0.117	70.70	72.70	71.700	0.139
1275	1.80	2.30	2.050	0.114	92.80	94.80	93.800	0.147
1300	0.75	1.00	0.875	0.115	116.40	118.40	117.400	0.161
1325	0.10	0.60	0.350	0.116	140.80	142.80	141.800	0.179
1350	0.15	0.50	0.325	0.132	165.50	167.50	166.500	0.198
1400	0.00	0.50	0.250	0.157	-	-	-	-
1500	0.00	0.50	0.250	0.213	-	-	-	-

$S_0 = 1183.74$

Source: Optionmetrics

Figure 1: Risk Neutral Distribution from Raw Options Prices

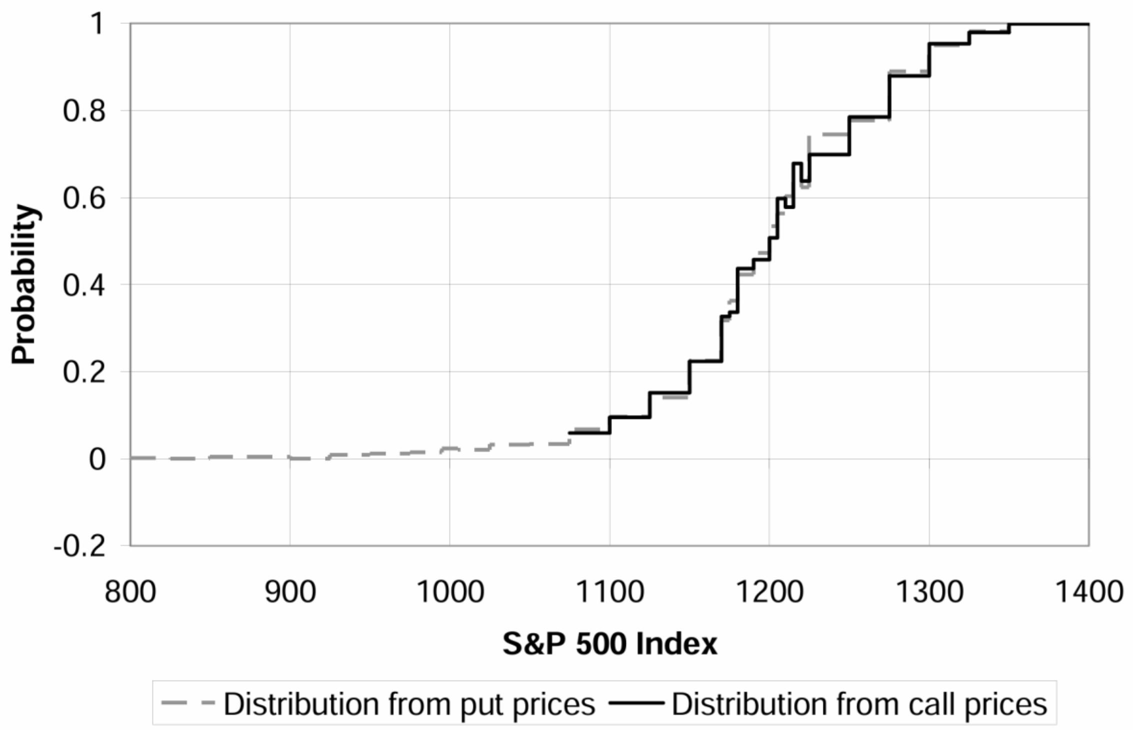
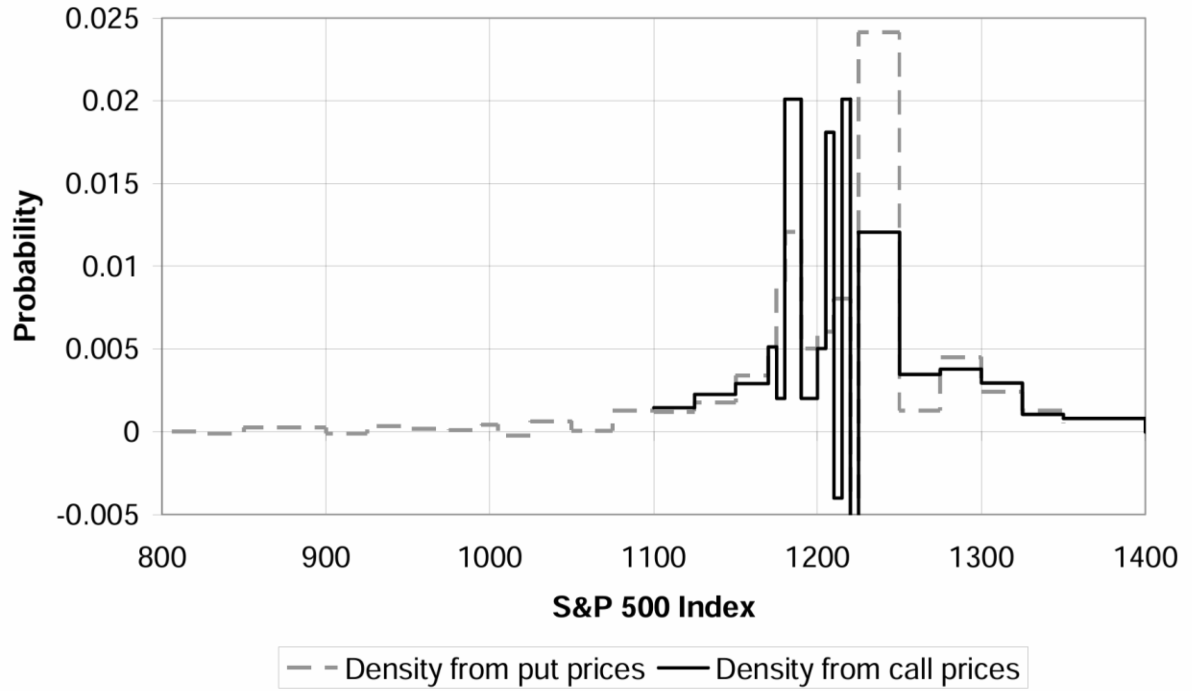
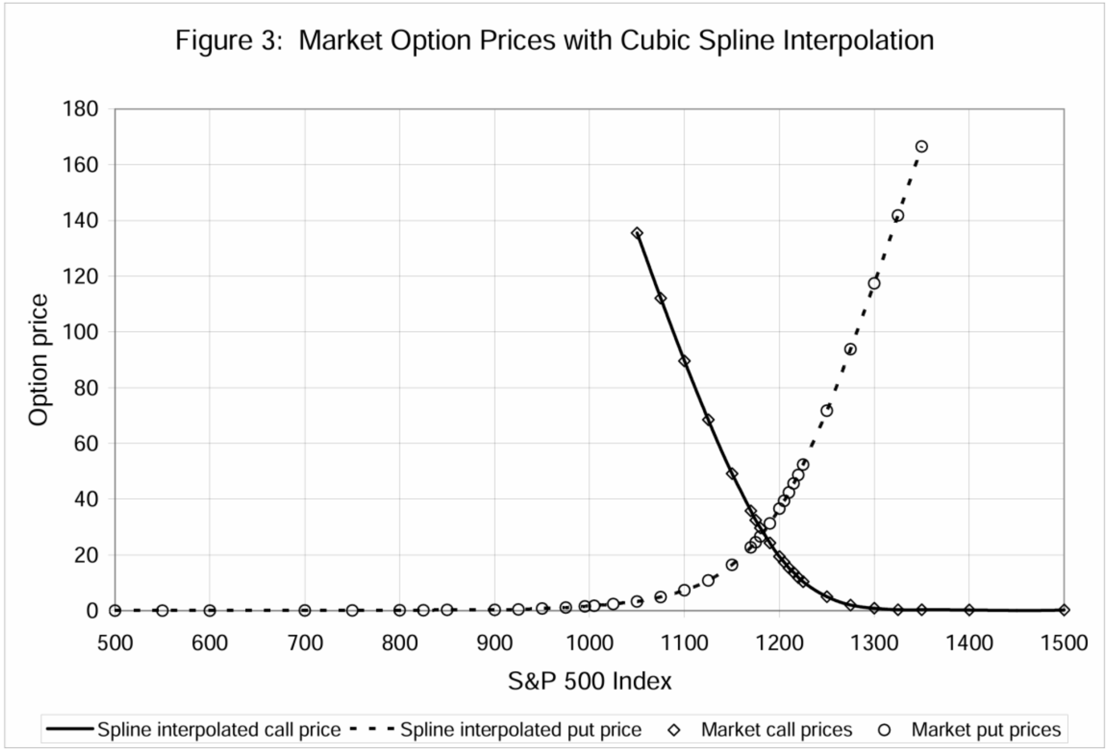


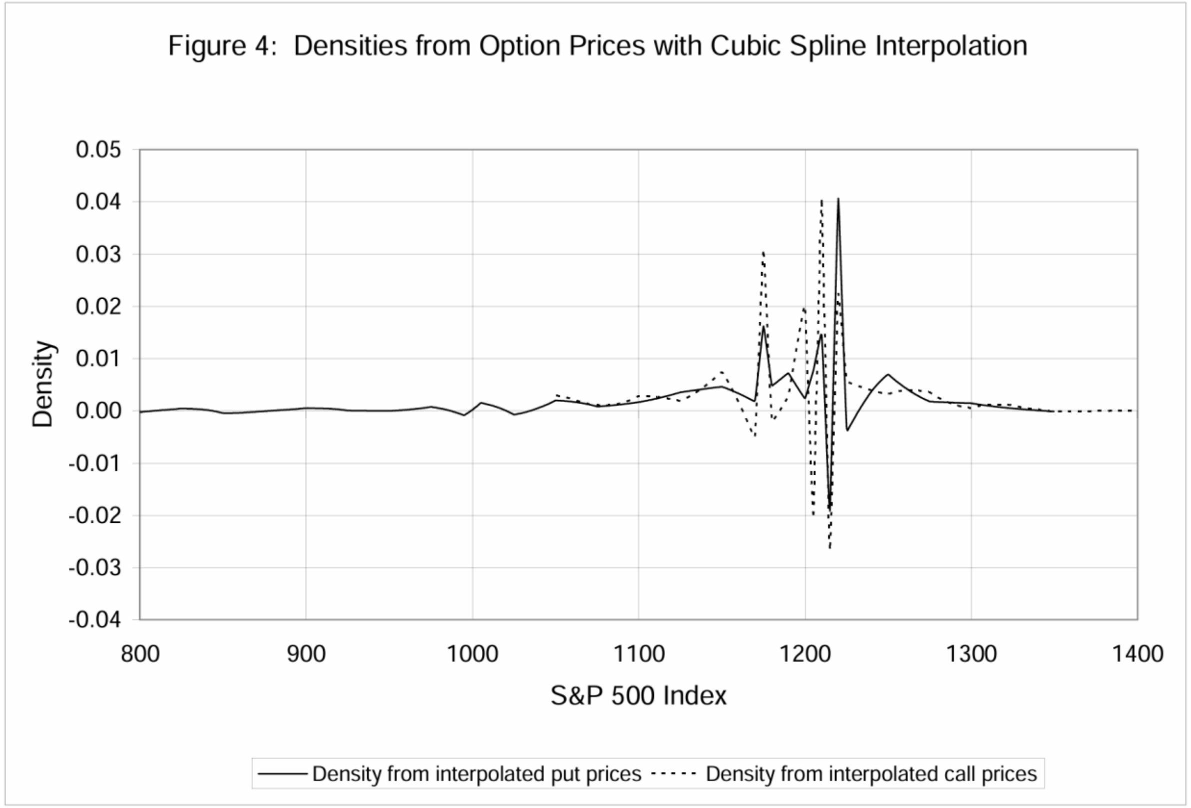
Figure 2: Risk Neutral Density from Raw Options Prices



This suggests the use of an interpolation technique to fill in intermediate values between the traded strike prices and to smooth out the risk neutral distribution.

Cubic spline interpolation is a very common first choice as an interpolation tool.





David Shimko (1993) proposed transforming the market option prices into implied volatility (IV) space before interpolating, then retransforming the interpolated curve back to price space to compute a risk neutral distribution.

Applying a cubic spline to interpolate the volatility smile still produces bad results for the fitted RND.

- The main reason for this is that an n-th degree spline constructs an interpolating curve consisting of segments of n-th order polynomials joined together at a set of "knot" points.
- At each of those points, the two curve segments entering from the left and the right are constrained to have the same value and the same derivatives up to order n-1.
- Thus a cubic spline has no discontinuities in the level, slope or 2nd derivative, meaning there will be no breaks, kinks, or even visible changes in curvature at its knot points.
- But when the interpolated IV curve is translated back into option strike-price space and the RND is constructed by taking the second derivative , the discontinuous 3rd derivative of the IV curve becomes a discontinuous first derivative in the RND.

The simple solution is just to interpolate with a 4th order spline or higher.

For the very deep in the money case, the effect of optionality is quite limited, such that the IV might range from 12.9% to 14.0% within the bid-ask spread. For the lowest strike call, there is no IV at the bid price, because it is below the no-arbitrage minimum call price. The IV at the ask is 15.6%, while the IV at the midpoint, which is what goes into the calculations, is 11.8%. In addition to the wide bid-ask spreads, there is little or no trading in deep in the money contracts. On this day, no 1050 or 1075 strike calls were traded at all, and only 3 1150 strike calls changed hands. **Most of the trading is in at the money or out of the money contracts.**

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Source: Optionmetrics

ITM (calls above S_0), OTM (puts above S_0)

$S_0 = 1183.74$

OTM (calls below S_0), ITM (puts below S_0)

But out of the money contracts present their own data problems, because of extremely wide bid-ask spreads relative to their prices. The 925 strike put, for example, would have an IV of 22.3% at its bid price of 0.20 and 26.2% at the ask price of 0.70. Setting the IV for this option at 24.8% based on the mid-price of 0.45 is clearly rather arbitrary.

To incorporate these ideas into our methodology, we first discard all options whose bid prices are less than 0.50

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ITM
OTM
$S_0 = 1183.74$
OTM
ITM

Blending the call and put IVs in the region around the at the money index level.

Next we want to combine calls and puts, using the out of the money contracts for each. But from Table 1, with the current index level at 1183.74, if we simply use puts with strikes up to 1180 and calls with strikes from 1190 to 1300, there will be a jump from the put IV of 14.2% to the call IV of 12.6% at the break point.

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OTM

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Blending the call and put IVs in the region around the at the money index level.

We have chosen a range of 20 points on either side of the current index value S0 in which the IV will be set to a weighted average of the IVs from the calls and the puts.

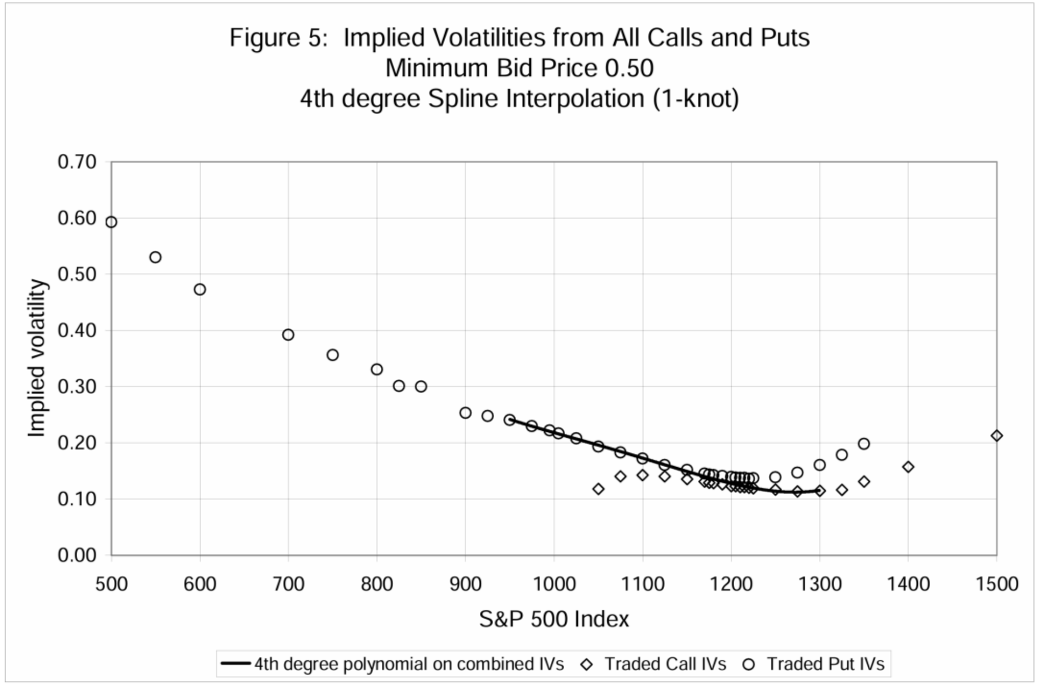
Let X_{low} be the lowest traded strike such that $(S_0 - 20) \leq X_{low}$ and X_{high} be the highest traded strike such that $X_{high} \leq (S_0 + 20)$. For traded strikes between X_{low} and X_{high} we use a blended value between $IV_{put}(X)$ and $IV_{call}(X)$, computed as

(10) $$IV_{blend}(X) = w\, IV_{put}(X) + (1-w)\, IV_{call}(X)$$

where $$w = \frac{X_{high} - X}{X_{high} - X_{low}}$$

Blending the call and put IVs in the region around the at the money index level.

In this case, we take put IVs for strikes up to1150, blended IVs for strikes 1170 to 1200, and call IVs for strikes from 1205 up.



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	1215	12.90	14.50	13.700	0.122	44.60	46.60	45.600	0.138	
	1220	11.10	12.70	11.900	0.120	47.70	49.70	48.700	0.136	ITM
OTM	**1225**	9.90	10.90	10.400	0.119	51.40	53.40	52.400	0.137	
	1250	4.80	5.30	5.050	0.117	70.70	72.70	71.700	0.139	
	1275	1.80	2.30	2.050	0.114	92.80	94.80	93.800	0.147	
	1300	0.75	1.00	0.875	0.115	116.40	118.40	117.400	0.161	
	1325	0.10	0.60	0.350	0.116	140.80	142.80	141.800	0.179	
	1350	0.15	0.50	0.325	0.132	165.50	167.50	166.500	0.198	
	1400	0.00	0.50	0.250	0.157	-	-	-	-	
	1500	0.00	0.50	0.250	0.213	-	-	-	-	

Source: Optionmetrics

Incorporating Market Bid-Ask Spreads

- 前面使用4th order spline擬合IV曲線時，會使用least square找出optimal parameters
- $Minimize\ \sum_i (IV_{observed,i} - IV_{spline,i})^2$→任何情況的偏差都equally weighted
- It would make sense to be more concerned about cases where the spline fell outside the quoted spread than those remained within it. → 懲罰在Bid-Ask Spread 外的$IV_{spline,i}$
- adapt the cumulative normal distribution function to construct a weighting function that allows weights between 0 to 1 as a function of a single parameter σ.

- $W_i(IV_{spline,i}) = \begin{cases} N[IV_{spline,i} - IV_{Ask,i}，\sigma] & if\ IV_{Midpoint} \leq IV_{spline,i} \\ N[IV_{Bid,i} - IV_{spline,i}，\sigma] & if\ IV_{spline,i} \leq IV_{Midpoint} \end{cases}$
- $if\ IV_{Midpoint} \leq IV_{spline,i}，N\left(\frac{IV_{spline,i} - IV_{Ask,i}}{\sigma}\right)$
- $if\ IV_{spline,i} \leq IV_{Midpoint}，N\left(\frac{IV_{Bid,i} - IV_{spline,i}}{\sigma}\right)$
- $Minimize\ \sum_i W_i (IV_{observed,i} - IV_{spline,i})^2$

Incorporating Market Bid-Ask Spreads

- $W_i(IV_{spline,i}) = \begin{cases} N\left[IV_{spline,i} - IV_{Ask,i}\ ,\ \sigma\right] & if\ IV_{Midpoint} \leq IV_{spline,i} \\ N\left[IV_{Bid,i}\ - IV_{spline,i}\ ,\ \sigma\right] & if\ IV_{spline,i}\ \leq IV_{Midpoint} \end{cases}$

- $if\ IV_{Midpoint} \leq IV_{spline,i}\ ,\ N\left(\frac{IV_{spline,i} - IV_{Ask,i}}{\sigma}\right)$

- $if\ IV_{spline,i}\ \leq IV_{Midpoint}\ ,\ N\left(\frac{IV_{Bid,i}\ - IV_{spline,i}}{\sigma}\right)$

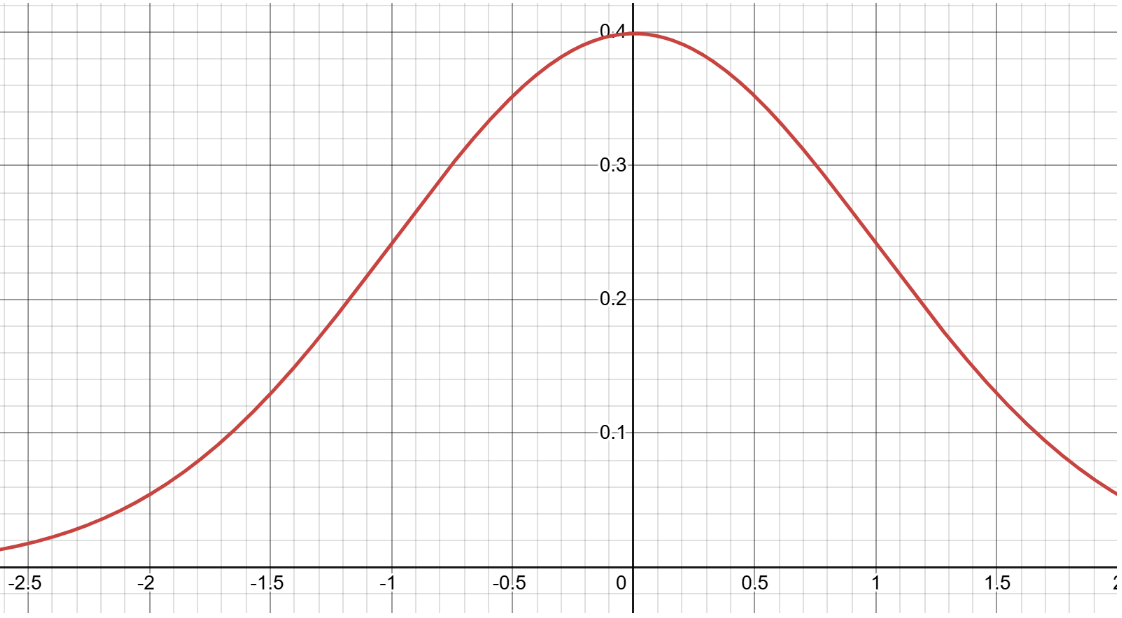
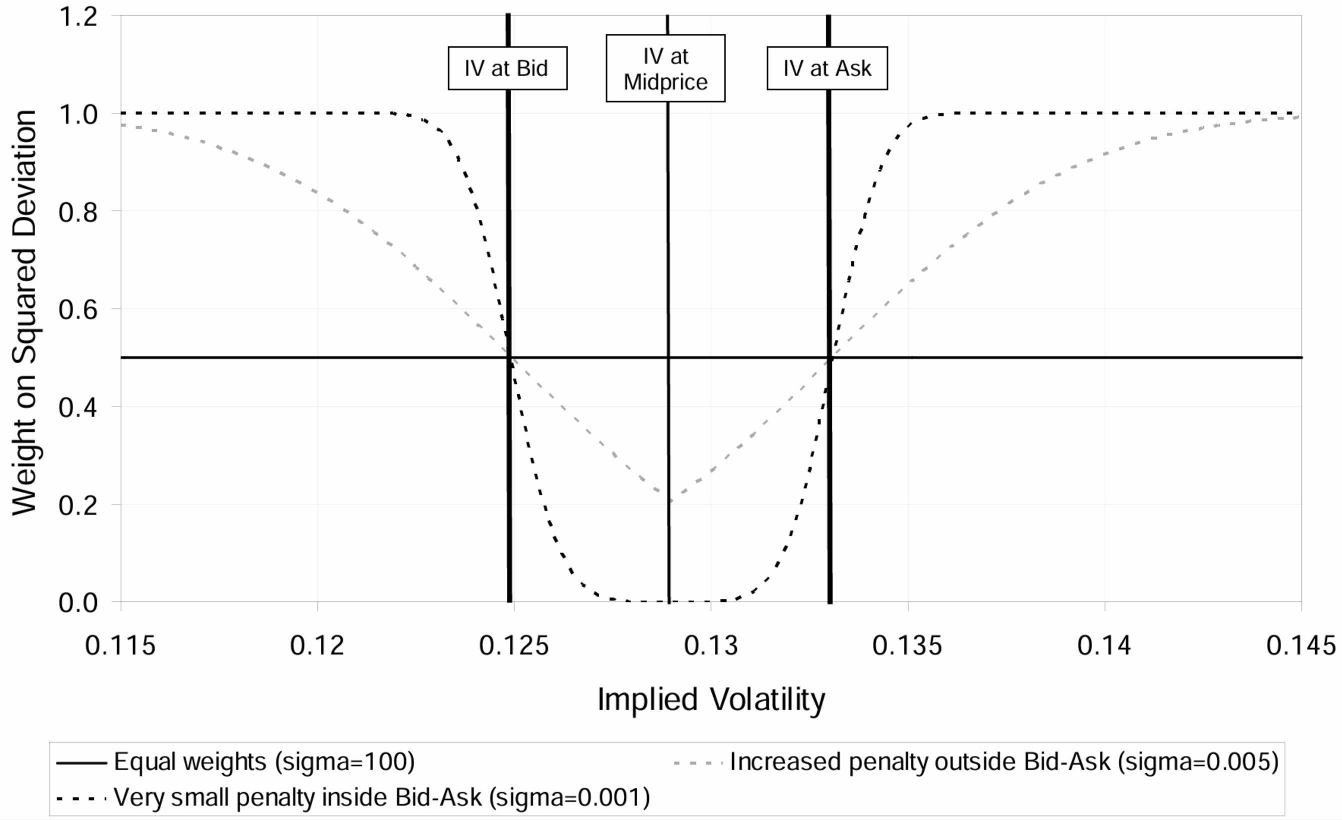


Figure 6: Alternative Weighting of Squared Deviations
Within and Outside the Bid-Ask Spread



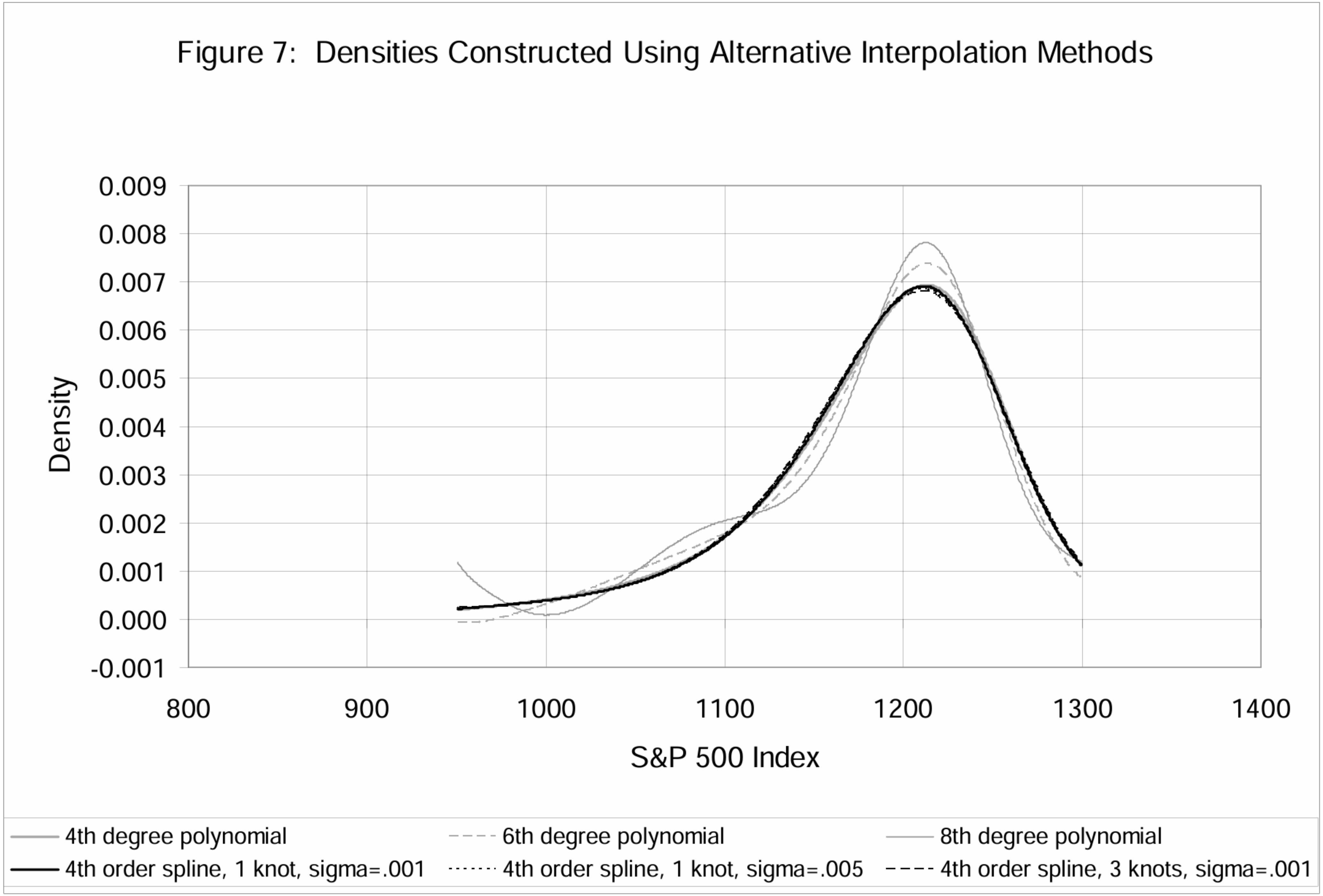


Figure 7: Densities Constructed Using Alternative Interpolation Methods
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0.000
-0.001
Density
800
900
1000
1100
1200
1300
1400
S&P 500 Index
4th degree polynomial
6th degree polynomial
8th degree polynomial
4th order spline, 1 knot, sigma=.001
4th order spline, 1 knot, sigma=.005
4th order spline, 3 knots, sigma=.001

1. Begin with bid and ask quotes for calls and puts with a given expiration date.
2. Discard quotes for very deep out of the money options. We required a minimum bid price of $0.50 for this study.
3. Combine calls and puts to use only the out of the money and at the money contracts, which are the most liquid.
4. Convert the option bid, ask and midprices into implied volatilities using the Black Scholes equation. To create a smooth transition from put to call IVs, take weighted averages of the bid, ask and midprice IVs from puts and calls in a region around the current at the money level.
5. Fit a spline function of at least 4th order to the midprice implied volatilities by minimizing the weighted sum of squared differences between the spline curve and the midprice IVs. The weighting function shown in equation (11) downweights deviations that lie within the market's quoted bid-ask spread relative to those falling outside it. The number of knots should be kept small, and their optimal placement may depend on the particular data set under consideration. In this study we used a 4th order spline with a single knot at the money.
6. Compute a dense set of interpolated IVs from the fitted spline curve and then convert them back into option prices.
7. Apply the procedure described in Section 3 to the resulting set of option prices in order to approximate the middle portion of the RND.

Adding Tails to the Risk Neutral Density

- Extending the empirical RND by grafting onto it tails drawn from a suitable parametric probability distribution in such a way as to match the shape of the estimated RND over the portion of the tail region for which it is available.
- **Fisher-Tippett Theorem**

  let $\boldsymbol{x_1}$**,** $\boldsymbol{x_2}$**, ... be an i.i.d. sequence of draws from some distribution F** and let M_n denote the maximum of the first n observations. If we can find sequences of real numbers a_n and b_n such that the sequence of normalized maxima $\frac{(\boldsymbol{M_n}-\boldsymbol{b_n})}{\boldsymbol{a_n}}$ **converges in distribution to some non-degenerate distribution H(x),** i.e.,

  P($\frac{(\boldsymbol{M_n}-\boldsymbol{b_n})}{\boldsymbol{a_n}}$ ≤ x) → H(x) as n → ∞ then H is a GEV distribution.

- **GEV distribution (CDF)**

  $F(z) = e^{-(1+\xi z)^{\frac{-1}{\xi}}}$ ，ξ determines the tail shape，$z = \frac{S_T - \mu}{\sigma}$

  - 當 ξ>0 時，表示極值具有重尾特性
  - 當 ξ=0 時，表示極值分布尾部呈指數衰減
  - 當 ξ<0 時，表示分布有一個有限的上界（或下界，取決於是否考慮最大值或最小值）。

Adding Tails to the Risk Neutral Density

GEV distribution (CDF)

$$F(z) = e^{-(1+\xi z)^{\frac{-1}{\xi}}}$$ ，ξ determines the tail shape ，$z = \frac{S_T - \mu}{\sigma}$

選擇連接點

- $X_1, X_2, \ldots, X_N$ are available strike price from the market
- $F_{EVL}(\cdot)$ and $F_{EVR}(\cdot)$ to denote the approximating CDF of GEV distributions for the left and right tails
- $F_{EMP}(\cdot)$ denotes the estimated empirical risk neutral distribution.
- $f_{EVL}(\cdot)$ and $f_{EVR}(\cdot)$ as the corresponding density functions
- $X(\alpha)$denotes the exercise price corresponding to the α-quantile of the risk neutral distribution. $F_{EMP}(X(\alpha)) = \alpha$

右尾, we choose α_{0R} and α_{1R} , there are some conditions for the right tail：

1. $X(\alpha_{1R}) \leq X_{N-1}$
2. $F_{EVR}(X(\alpha_{0R})) = \alpha_{0R}$
3. $f_{EVR}(X(\alpha_{0R})) = f_{EMP}(X(\alpha_{0R}))$
4. $f_{EVR}(X(\alpha_{1R})) = f_{EMP}(X(\alpha_{1R}))$

The GEV parameter values that will cause these conditions to be satisfied can be found easily using standard optimization procedures.

Adding Tails to the Risk Neutral Density

GEV distribution (CDF)

$$F(z) = e^{-(1+\xi z)^{\frac{-1}{\xi}}}$$ ， ξ determines the tail shape

左尾, GEV本身是用來描述「最大值」的分布，要描述最小值的分布需要將S_T加上一個負號，將極小值變為極大值

$$z = \frac{-S_T - \mu_L}{\sigma} = \frac{-\mu_L - S_T}{\sigma}$$ ， μ_L is a positive value

For left tail, we choose α_{0L} and α_{1L} , there are some conditions for the right tail：

1. $X_2 \leq X(\alpha_{1L})$
2. $F_{EVL}(-X(\alpha_{0L})) = 1 - \alpha_{0L}$
3. $f_{EVL}(-X(\alpha_{0L})) = f_{EMP}\big(X(\alpha_{0L})\big)$
4. $f_{EVL}(-X(\alpha_{1L})) = f_{EMP}(X(\alpha_{1L}))$

The GEV parameter values that will cause these conditions to be satisfied can be found

easily using standard optimization procedures.

Figure 8: Risk Neutral Density and Fitted GEV Tail Functions